Profinite Detection of Free Factors

Mathematics in Conversation
3 March, 2026

Broad Question How much information on an infinite group G survives on it's finite quotients?

Specific Question Can the set of finite quotients of an infinite group G tell if G splits as a free product of two subgroups?

§ 1 Free products

Philosophy of gp. theory:

Take a big, complicated group, break it into irreducible parts & study these simpler atoms + how they fit together.

Simplest way of glueing 2 groups:

direct products!

H, K groups $\quad H \times K = \{(h,k) : h \in H, k \in K\}$.
entrywise operation

⇝ universal "go in" property

$$\begin{tikzcd} & H & \\ Q \arrow[ur, "f_H"] \arrow[rr, dashed, "f"] \arrow[dr, "f_K"'] & & H \times K \arrow[ul] \arrow[dl] \\ & K & \end{tikzcd}$$

$f(q) = (f_H(q), f_K(q))$

⇝ Examples

- F.g. abelian groups

$$\mathbb{Z}\times\mathbb{Z}\times\dots\times\mathbb{Z}\times C_{m_1}\times C_{m_2}\times\dots\times C_{m_k}$$

- $\pi_1(X\times Y) = \pi_1(X)\times\pi_1(Y)$

- S compact surface, $\pi_1(S) = H\times K$
  $\Leftrightarrow$ $S =$  , $H, K = \mathbb{Z}$

- M closed 3 manifold, $\pi_1(S) = H\times K$
  $\Leftrightarrow$ $M = S\times S^1$, $H = \pi_1(S)$, $K = \mathbb{Z}$.

⇝ Atomic parts

Directly indecomposable groups:
$\mathbb{Z}$, $\pi_1(S)$ for $S \neq$ ,
C_p, S_n, $SL_2\mathbb{Z}$ ...

⚠ No uniqueness of decomposition
($\exists\ H \not\cong K$ s.t. $H\times\mathbb{Z} \cong K\times\mathbb{Z}$)

⇝ Inner criterion

If $H, K \leq G$ are such that

- $H\cap K = \{1\}$
- $gHg^{-1} = H$, $gKg^{-1} = K$
- $HK = G$

Then $G \cong H\times K$.

Free products

$H * K = \{ h_1 k_1 h_2 k_2 \cdots h_\ell k_\ell : \begin{matrix} h_i \in H \\ k_i \in K \end{matrix}, \ \ell \geq 0 \} / \sim$

↳ operation is concatenation + reduction

"Free" as in free of extra relations.

⇝ universal "go out" property

$$\begin{tikzcd} & H \arrow[dl, "f_H"'] \arrow[dr] & \\ Q & & H * K \arrow[ll, dashed, "f"'] \\ & K \arrow[ul, "f_K"] \arrow[ur] & \end{tikzcd}$$

$f(h_1 k_1 \cdots h_\ell k_\ell)$
$= f_H(h_1) f_K(k_1) \cdots f_H(h_\ell) f_K(k_\ell)$

⇝ Examples

- Free groups: $F_m = \underbrace{\mathbb{Z} * \mathbb{Z} * \cdots * \mathbb{Z}}_{m \text{ times}}$

- $\pi_1(X \vee Y) = \pi_1(X) * \pi_1(Y)$

- $D_\infty = C_2 * C_2$, $\quad PSL_2 \mathbb{Z} = \overset{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}{C_2} * \overset{\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}}{C_3}$

- $\pi_1(S) = H * K \Leftrightarrow \partial S \neq \emptyset \Leftrightarrow \pi_1(S)$ is free

- If M is a closed 3-manifold & $M = M_1 \# \cdots \# M_k$ (connected sum), then $\pi_1(M) = \pi_1(M_1) * \cdots * \pi_1(M_k)$.

⇝ Atomic parts

Freely indecomposable groups.

Theorem (Grushko, 1940). If G is f.g., then

$$G \cong G_1 * \cdots * G_s * F_m$$

where F_m is free of rank m, each G_i is not cyclic & freely indecomposable. More, m & s are unique, and so are the G_i up to order and conjugacy.

⇝ Inner criterion

⚠ None, really :-(

⇝ Alternatives.

① Trees: Bass - Serre theory

X oriented graph, VX vertices, EX edges
$\partial_0, \partial_1 : EX \to VX$.

G group, $G \curvearrowright X$ means $g \mapsto \sigma_g$

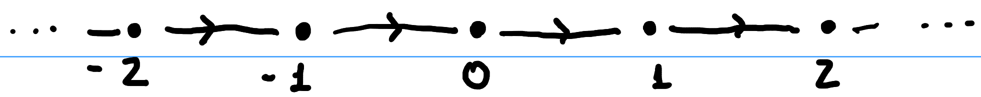
$$\sigma_g : X \to X,$$

$$\sigma_g(VX) \subseteq VX, \quad \sigma_g(EX) \subseteq EX, \quad \sigma_1 = id, \quad \sigma_{gh} = \sigma_g \circ \sigma_h,$$

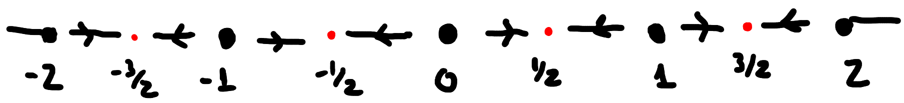
$$\partial_i \circ \sigma_g = \sigma_g \circ \partial_i .$$

$$\mathrm{Stab}_G(x) = \{ g \in G : \sigma_g(x) = x \} \leqslant G$$

Example D_∞ = isometries of $\mathbb{Z}$ = {translation ∘ reflection}



⚠ does not preserve orientation. Solution:



$\mathrm{Stab}_{D_\infty}(\dot{m}) = \{id, \text{ reflection around } m\}$

$\mathrm{Stab}_{D_\infty}(\dot{m+1/2}) = \{id, \text{ reflection around } m \circ \text{translation by} -1\}$

$\mathrm{Stab}_{D_\infty}(\rightarrow) = \{id\}$

Theorem (Serre, 1977) $G = H * K \Leftrightarrow \exists$ tree X s.t. $G \curvearrowright X$ & (1) $G \backslash X$ is a segment, (2) $\mathrm{Stab}_G(v)$ conjugate to H or K $\forall v \in VX$, (3) $\mathrm{Stab}_G(e) = \{1\}$ $\forall e \in EX$.

Example Standard tree X of $G = H * K$

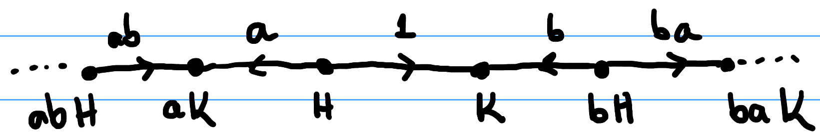
Vertices: $VX = G/H \cup G/K$

$EX = G$

$\partial_0(g) = gH$, $\quad \partial_1(g) = gK$

If $H = C_2 = \langle a \rangle$ & $K = C_2 = \langle b \rangle$,

$G = \{ababa\cdots, babab\ldots\}$



② Derivations

$\mathbb{F}_2 = \{0,1\}$ field of two elements.

$\mathbb{F}_2 G$ = group algebra, $\mathbb{F}_2$-vector space with basis G, multiplication extended linearly from G.

$G \curvearrowright \mathbb{F}_2 G$ by left multiplication

Def A derivation $d : G \longrightarrow \mathbb{F}_2 G$ is a function satisfying
$$d(gh) = d(g) + g\, d(h).$$

Example • For any $x \in \mathbb{F}_2 G$,
$$ad_x(g) = gx - x \in \mathbb{F}_2 G$$
is a derivation. Derivations of the form ad_x are called inner derivations.

• Let $G = H * K$ and X be its standard tree: $VX = G/H \cup G/K$, $EX = G$, $\partial_0(g) = gH$, $\partial_1(g) = gK$. Choose any vertex $v \in VX$, and define
$$d(g) = \sum_{h \in [v, gv]} h \in \mathbb{F}_2 G$$
where $[v, gv]$ is the geodesic between v & gv. Then, d is a derivation. If $H, K \neq 1$, it is not inner.

Theorem (Dunwoody, 1979) Let G be a group without elements of finite order. Then G decomposes as a free product if and only if there exists a non-inner derivation $d : G \longrightarrow \mathbb{F}_2 G$.

§2. Profinite properties.

Let G be a group and define:

$C(G) = \{ G/N : N \trianglelefteq G,\ G/N \text{ finite} \}$.

Broad question What does $C(G)$ know about G?

A property of G is called profinite if it can be read of from $C(G)$.

Examples Consider the group $(\mathbb{Q}, +)$.

$$\mathbb{Q}/N \text{ finite} \Rightarrow \mathbb{Q}/N = \{0\}.$$

Hence, $C(\mathbb{Q}) = C(\{0\}) = \{\{0\}\}$. Same for $\mathbb{R}$.
Finite quotients can't tell apart $\mathbb{Z}$ & $\mathbb{Z} \times \mathbb{Q}$!

Def A group G is residually finite (RF) if $\forall g \in G$ non-trivial exists $N \trianglelefteq G$ with G/N finite such that $gN \neq N$.

Profinite properties only make sense for RF groups.

Example • Is RF:

- $\mathbb{Z}$, finite gps, f.g. abelian gps
- f.g. subgroups of $GL_n \mathbb{C}$
- D_∞, $PSL_2 \mathbb{Z}$
- Free groups
- $\pi_1(S)$ for S compact surface.
- $\pi_1(M)$ for M closed 3-manifold
- H, K RF $\Rightarrow$ $H \times K$ & $H * K$ RF

• Is not RF

- $\mathbb{Q}, \mathbb{R}, \mathbb{C}, \mathbb{Q}/\mathbb{Z}, S^1$

* any non-trivial divisible gp
* Canonical f.g. example: let $n, m \neq 0$ be integers and define:

$$BS(n,m) = \langle a,b \mid ba^n b^{-1} = a^m \rangle$$
$$= \text{Free gp on } \{a,b\} \,/\, \text{normal subgroup gen. by } ba^n b^{-1} a^{-m}.$$

$BS(n,m)$ is RF iff

* $|n| = 1$, or
* $|m| = 1$, or
* $|n| = |m|$.

$BS(2,3)$ is the first non-residually finite.

Facts If G is f.g. RF, then $C(G)$ knows if:

* G is abelian, nilpotent, solvable ...
* if G is abelian and $C(G) = C(H)$, $G \cong H$.
* if G is nilpotent, there is a finite list $H_1, \ldots, H_k$ of gps up to iso s.t. $C(G) = C(H_i)$.

Open problem (Remeslennikov 1976) If G is f.g. RF & $C(G) = C(F_n)$, is $G \cong F_n$?

Grothendieck If G is "very far" from abelian, $C(G)$ should determine $G \Rightarrow$ anabelian geometry.

Put the discrete topology on each G/N and the product topology on

$$\prod_{C(G)} G/N.$$

We have a map $G \longrightarrow \prod_{C(G)} G/N$ given by $g \mapsto (gN)$.

Denote by $\hat{G}$ the closure of the image of G in $\prod G/N$. Then $\hat{G}$ is a compact, Hausdorff totally disconnected topological group.

Prop. $G \to \hat{G}$ is injective iff G is RF. If G, H are f.g. RF groups, then $C(G) = C(H)$ iff $\hat{G} \cong \hat{H}$ as topological groups.

$\hat{G}$ is called the profinite completion of G. It has same information as $C(G)$, but more structure.

Examples If $G = H \times K$, then $\hat{G} \cong \hat{H} \times \hat{K}$ as top. groups. ($\Leftarrow$) False in general! $\exists$ f.g. R.F groups G that are directly indecomposable but $\hat{G} = A \times B$.

Def A profinite gp is a Compact, Hausdorff totally disconnected top. gp.

⇝ infinite + profinite ⇒ uncountable, massive!

⇝ direct product of profinite gps is profinite

⇝ No clear topology on the free product of profinite groups. How to define profinite free product?

If A, B are profinite & top. finitely generated:

$$A \amalg B = \widehat{A * B}.$$

⇝ Not all elements of $A \amalg B$ are words.

~> Universal "go out" property

$$
\begin{tikzcd}
& A \arrow[dl] \arrow[dr] & \\
C & & A \amalg B \arrow[ll, dotted] \\
& B \arrow[ul] \arrow[ur] &
\end{tikzcd}
$$

~> Examples $G = H * K$, then $\widehat{G} = \widehat{H} \amalg \widehat{K}$.
($\Leftarrow$) widely open.

~> Atomic parts: freely indecomposable profinite gps

⚠ Uniqueness & finiteness of decomposition does not hold! (Lucchini 2001)

~> Bass - Serre

Kinda works? → trees become profinite "trees"
→ action must be continuous

Theorem $C = A \amalg B \Leftrightarrow$ "profinite tree" X and continuous action $C \curvearrowright X$ s.t. (1) $G \backslash X$ is a segment, (2) $\mathrm{Stab}_C(v)$ conjugate to A or B $\forall v \in VX$, $\mathrm{Stab}_C(e) = \{1\}$ $\forall e \in EX$.

Standard profinite tree X:
$VX = C/A \sqcup C/B$
$EX = C$
$\partial_0(g) = gA$, $\partial_1(g) = gB$

Not "connected" but topologically connected.
No finite nor infinite cycles.

~> Derivations AFAIK, only one way.

$C = A \amalg B$ $\xrightarrow{\text{exists!}}$ $d : C \to \widehat{\mathbb{F}_2 C}$ non inner

↪ closure of $\mathbb{F}_2 C$ in $\prod \mathbb{F}_2(C/N)$

big open problem

C(G) detects free products

Main questions 1) if $\hat{G} = A \amalg B$, $G = H * K$ for some H, K?

2) If for $H \leq G$ exists $B \leq \hat{G}$ s.t. $\hat{G} = \hat{H} \amalg B$, must we have $G = H * K$ for some $K \leq G$?

C(G) detects free factors

§ 3 Recent progress

Difficulties:

- Standard profinite tree is not a tree, uncountable + highly disconnected.
- $\widehat{\mathbb{F}_2 \hat{G}}$ much bigger than $\mathbb{F}_2 G$.
$$d: \hat{G} \longrightarrow \widehat{\mathbb{F}_2 \hat{G}}$$ may not restrict to $$d: G \to \mathbb{F}_2 G.$$
- Even if it restricts, it may be inner!

Consider the class VF of groups G of the form:

$$G = Q_1 * Q_2 * \cdots * Q_k * F_m$$

free gp

finite gps

Such G are f.g. RF.

Theorem (Garrido - Jaikin, 23) C(G) detects free factors. for G in the class VF.

Key lemma For any G in VF and any finite set $x_1, \dots, x_k \in \widehat{\mathbb{F}_2 \hat{G}}$, there exists a G-invariant subspace $M \leq \widehat{\mathbb{F}_2 \hat{G}}$ containing $x_1, \dots, x_k$ s.t.

$$M \cong \underbrace{\mathbb{F}_2 G \oplus \mathbb{F}_2 G \oplus \cdots \oplus \mathbb{F}_2 G}_{\text{finitely many copies}}.$$

- from $d: \hat{G} \longrightarrow \widehat{\mathbb{F}_2 \hat{G}}$, we get $d: G \longrightarrow M \simeq \mathbb{F}_2 G^{\ell}$.
- carefully build some $d': G \longrightarrow \mathbb{F}_2 G$ out of d.
  ↳ ensure it's not inner!

Until recently, this was state of the art. We expand this to the class D_1:
groups G that act on a tree X with vertex stabilizers in VF and edge stabilizers infinite cyclic.

↳ big jump!
contains $\pi_1(S)$ for compact surfaces, $\pi_1(M)$ for various compact 3-manifolds M with boundary, $BS(m,m)$ ....

<u>Theorem</u> (S. - Jaikin - Zalesski, submitted) Let G be a f.g. group from the class D_1. Assume that G does not contain $BS(n,m)$ for $|n| \neq |m|$. Then:

a) $\mathcal{C}(G)$ detects free products

b) $\mathcal{C}(G)$ detects freely indecomposable free factors.